

Area under a graph

The area under a graph often gives useful information.

Velocity-time graphs

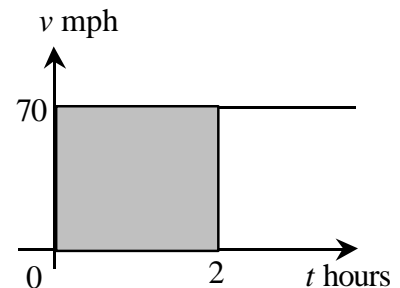
Constant velocity

The sketch shows the velocity-time graph for a car that is travelling along a motorway at a steady 70 mph.

The area under this graph is rectangular in shape.

The shaded area = $2 \times 70 = 140$

This is the distance in miles travelled in 2 hours when the speed is 70mph.



Area under a velocity-time graph = distance travelled.

Constant acceleration

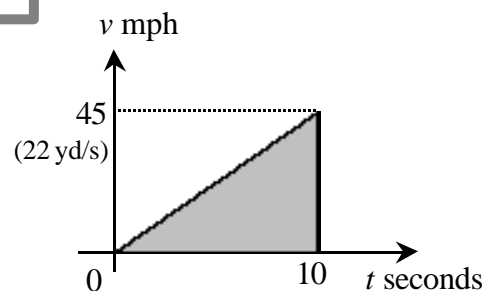
Now consider the case when a car is accelerating steadily from 0 to 45 mph in 10 seconds.

The graph shows this situation, but note that the velocity is in miles per hour whilst the time is in seconds. The units need to be converted in order to find the distance travelled.

$$45 \text{ mph} = \frac{45 \times 1760}{60 \times 60} = 22 \text{ yards per second.}$$

The distance travelled is given by the area under the graph:

$$\text{Distance travelled} = \frac{10 \times 22}{2} = 110 \text{ yards.}$$



(1 mile = 1760 yards)

Note that this is equivalent to the car travelling for 10 seconds at the average speed of 11 yards per second.

In each case sketch a velocity-time graph and find the distance travelled in miles or yards:

- 1 Car travels at a steady velocity of 65 mph for 3 hours.
- 2 Car decelerates steadily from a velocity of 36 mph until it stops 5 seconds later.
- 3 Car accelerates steadily from 30 mph to 60 mph over a period of 6 seconds.
- 4 Car decelerates steadily from 70 mph to 40 mph over a period of 9 seconds.



Variable velocity and acceleration

The table and graph give the velocity of a car as it travels between 2 sets of traffic lights.

<i>t</i> (s)	0	2	4	6	8	10	12
<i>v</i> (ms ⁻¹)	0	5	8	9	8	5	0

An *estimate* for the area under this graph can be found by splitting it into strips as shown.

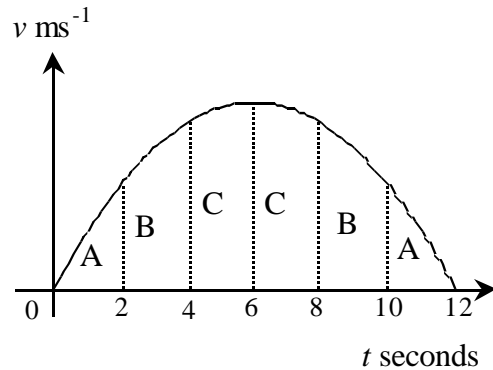
The strips at each end are approximately triangular in shape and each strip between them is approximately in the shape of a trapezium.

$$\text{Area of A} = \frac{2 \times 5}{2} \approx 5$$

$$\text{Area of B} = \frac{2(5 + 8)}{2} \approx 13$$

$$\text{Area of C} = \frac{2(8 + 9)}{2} \approx 17$$

$$\begin{aligned} \text{Total area} &\approx 5 + 13 + 17 + 17 + 13 + 5 \\ &= 70 \end{aligned}$$



This is a simplified model of this situation. In practice the change in velocity is unlikely to be so smooth and symmetrical.

This is an estimate of the distance travelled by the car (in metres) between the 2 sets of traffic lights.

Note that better estimates can be found by using more data and narrower strips.

In each of the following:

- draw a velocity-time graph on a graphic calculator or spreadsheet
- describe what happens during the given time interval
- estimate the distance travelled

1

<i>t</i> (s)	0	1	2	3	4	5	6	7	8
<i>v</i> (ms ⁻¹)	0	5.0	7.1	8.7	10.0	11.2	12.2	13.2	14.1

2

<i>t</i> (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
<i>v</i> (ms ⁻¹)	5.0	6.9	8.6	10.1	11.4	12.5	13.4	14.1	14.6	14.9	15.0

3

<i>t</i> (s)	0	2	4	6	8	10
<i>v</i> (ms ⁻¹)	25	16	9	4	1	0

4

<i>t</i> (s)	0	1	2	3	4	5	6	7	8	9
<i>v</i> (ms ⁻¹)	20.0	14.0	11.5	9.6	8.0	6.6	5.3	4.1	3.0	2.0



Integration

The area under a graph between $x = a$ and $x = b$ is:

$$A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y \delta x = \int_a^b y dx$$

This means the limit of the sum of rectangles of area $y \delta x$ as δx tends to zero.

The increment of area $\delta A = y \delta x$

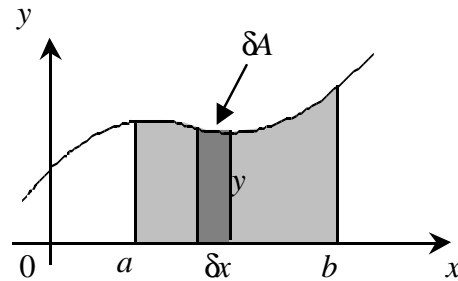
This can be written as $y = \frac{\delta A}{\delta x}$.

The limit of $\frac{\delta A}{\delta x}$ as δx tends to zero is $\frac{dA}{dx}$

i.e. the derivative of A with respect to x .

So $A = \int_a^b y dx$ is equivalent to $y = \frac{dA}{dx}$

If the area under a graph is divided into very narrow strips, each strip is approximately rectangular in shape.



i.e. the gradient of a graph of A against x .

Integration is the inverse of differentiation.

Rules of Integration

Reversing differentiation gives the rules shown:

eg $y = 5x$ gives $\frac{dy}{dx} = 5$

so $\int 5 dx = 5x$

In general, for any constant k , $\int k dx = kx$

Also $y = \frac{1}{2}x^2$ gives $\frac{dy}{dx} = x$

so $\int x dx = \frac{1}{2}x^2$

and $y = \frac{1}{3}x^3$ gives $\frac{dy}{dx} = x^2$

so $\int x^2 dx = \frac{1}{3}x^3$

In general, $\int x^n dx = \frac{x^{n+1}}{n+1}$

The derivative of a constant is zero. This means that when a constant is differentiated it disappears. When integrating a constant appears – this is called the **constant of integration**.

Expression	Integral
k (constant)	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^3	$\frac{x^4}{4} + c$
x^4	$\frac{x^5}{5} + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
kx^n	$\frac{kx^{n+1}}{n+1} + c$
$6x^2 - 2x + 1$	$\frac{6x^3}{3} - \frac{2x^2}{2} + x + c$ $= 2x^3 - x^2 + x + c$

where c is the constant of integration.

Because the value of the constant is not known (unless more information is given), these are known as **indefinite integrals**



Finding areas by integration

The area under a graph of a function is found by subtracting the value of the integral of the function at the lower limit from its value at the upper limit. The example below shows the method and notation used.

Example Find the area under the curve
 $y = 2x^3 - 9x^2 + 12x + 6$
 between $x = 1$ and $x = 3$

$$\begin{aligned}
 A &= \int_1^3 (2x^3 - 9x^2 + 12x + 6) dx \\
 &= \left[\frac{2x^4}{4} - \frac{9x^3}{3} + \frac{12x^2}{2} + 6x \right]_1^3 \\
 &= \left[\frac{x^4}{2} - 3x^3 + 6x^2 + 6x \right]_1^3 \\
 &= \left[\frac{3^4}{2} - 3 \times 3^3 + 6 \times 3^2 + 6 \times 3 \right] \\
 &\quad - \left[\frac{1^4}{2} - 3 \times 1^3 + 6 \times 1^2 + 6 \times 1 \right] \\
 &= [40.5 - 81 + 54 + 18] - [0.5 - 3 + 6 + 6]
 \end{aligned}$$

$$A = 22$$

The shaded area is 22 square units.

In each case:

- use a graphic calculator or spreadsheet to draw a graph of the function
- integrate to find the area under the graph between the given values of x .

1 $y = 3x^2$ between $x = 1$ and $x = 2$

2 $y = 2x^3$ between $x = 0$ and $x = 2$

3 $y = x^2 + 4$ between $x = 0$ and $x = 3$

4 $y = 6x - x^2$ between $x = 0$ and $x = 6$

5 $y = x^3 + 1$ between $x = 2$ and $x = 4$

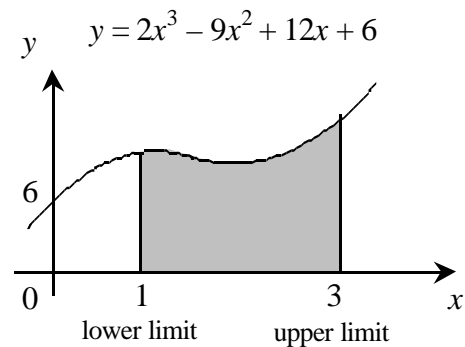
6 $y = 4x^2 - x + 4$ between $x = 1$ and $x = 3$

7 $y = x^3 - 4x^2 + 4x$ between $x = 0$ and $x = 2$

8 $y = 32 - x^5$ between $x = 0$ and $x = 2$

9 $y = \frac{10}{x^2}$ between $x = 1$ and $x = 5$

10 $y = \sqrt{x}$ between $x = 1$ and $x = 4$



The sketch shows the area required.

The constant of integration has been omitted because it would be in both parts – it always disappears when the parts are subtracted, so is normally left out of the working.

Because integration with limits does not give a result involving an unknown constant, it is known as **definite integration**.

Integration can be used to find the area of any shape as long as its boundaries can be written as functions of x .



More about velocity-time graphs

In velocity-time graphs the area gives the distance travelled. The following examples rework some of the previous examples using integration.

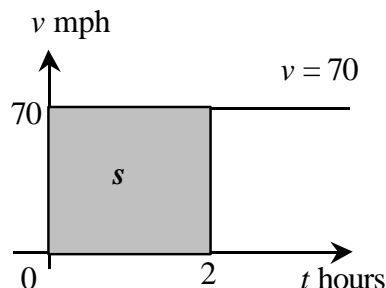
Example

Find the distance travelled when a car travels at a constant velocity of 70 mph for 2 hours.

$$\begin{aligned} s &= \int_0^2 70 \, dt = [70t]_0^2 \\ &= [70 \times 2] - [70 \times 0] = 140 \end{aligned}$$

The car travels **140 miles**.

In this case v is used instead of y and t instead of x . The distance is s .



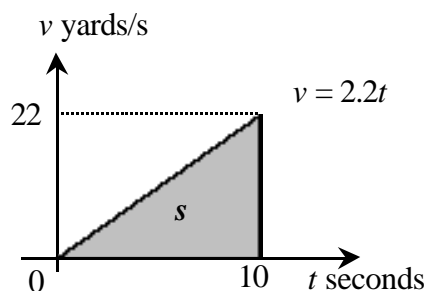
Example

Find the distance travelled when a car accelerates steadily to a velocity of 22 yards per second in 10 seconds.

$$\begin{aligned} s &= \int_0^{10} 2.2t \, dt = \left[\frac{2.2t^2}{2} \right]_0^{10} = [1.1t^2]_0^{10} \\ &= [1.1 \times 10^2] - [1.1 \times 0^2] = 110 \end{aligned}$$

The car travels **110 yards**.

(intercept = 0, gradient = 2.2)



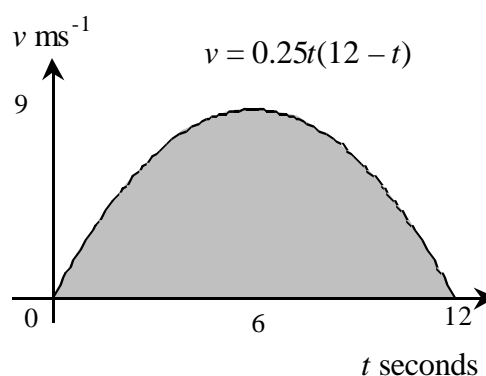
Example

The velocity of a car as it travels between two sets of traffic lights is modelled by $v = 0.25t(12 - t)$ where v is the velocity in metres per second and t is the time in seconds.

Find the distance travelled.

$$\begin{aligned} s &= \int_0^{12} (3t - 0.25t^2) \, dt \\ &= \left[\frac{3t^2}{2} - \frac{0.25t^3}{3} \right]_0^{12} \\ &= \left[\frac{3 \times 12^2}{2} - \frac{0.25 \times 12^3}{3} \right] - [0] = 72 \end{aligned}$$

The car travels **72 metres**.



Compare the answers from these examples with those given earlier. Use integration to check your answers to the questions at the bottom of page 1.



The functions given below can be used to model the velocity v metres per second at time t seconds for the examples given in questions 1 to 4 at the bottom of page 2.

In each case:

- *use a graphic calculator or spreadsheet to draw the graph of the model and compare the result with the graph you drew earlier*
- *use integration to estimate the distance travelled over the given time interval and compare the result with your earlier answer.*

1 $v = 5\sqrt{t}$ between $t = 0$ and $t = 8$

2 $v = 5 + 4t - 0.4t^2$ between $t = 0$ and $t = 5$

3 $v = \frac{1}{4}t^2 - 5t + 25$ between $t = 0$ and $t = 10$

4 $v = 20 - 6\sqrt{t}$ between $t = 0$ and $t = 9$

Each of the following functions models the velocity v metres per second in terms of the time t seconds. In each case:

- *use a graphic calculator or spreadsheet to draw the graph of the model*
- *briefly describe how the velocity varies over the given time interval*
- *integrate to find the distance travelled between the given values of t .*

5 $v = 0.15t^2$ between $t = 0$ and $t = 8$

6 $v = 8 - t^3$ between $t = 0$ and $t = 2$

7 $v = t^2 - 4t + 4$ between $t = 0$ and $t = 4$

8 $v = 2 - 0.5\sqrt{t}$ between $t = 0$ and $t = 16$



Teacher Notes

Unit Advanced Level, Modelling with calculus

Skills used in this activity:

- finding areas under graphs
- estimating areas using triangles and trapezia
- integrating polynomial functions

Preparation

Students need to know how to use a graphc calculator or spreadsheet to draw graphs from a set of data values or given functions and how to find the area of rectangles, triangles and trapezia.

Notes on Activity

This activity uses areas under velocity-time graphs to introduce integration.

The main points are also included in the Powerpoint presentation of the same name.

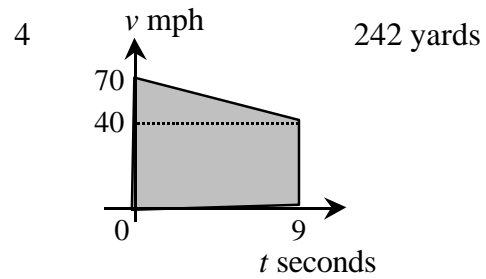
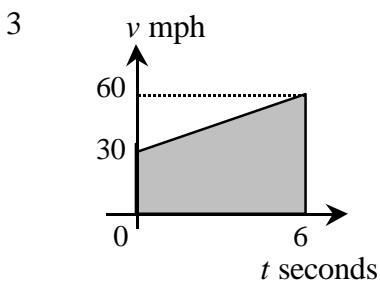
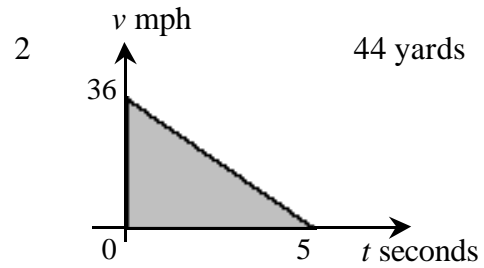
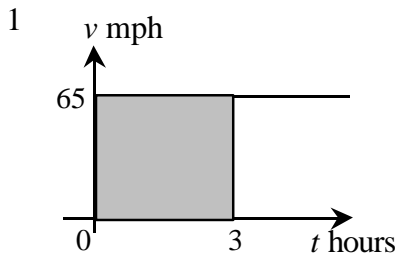
If your students have calculators that can integrate, this method could be used to check their answers.

The answers to the questions, including sketches of the graphs, are given on the following two pages.

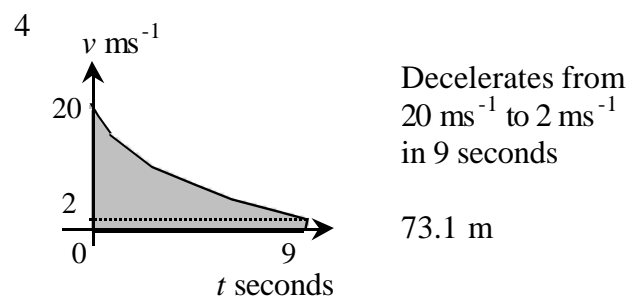
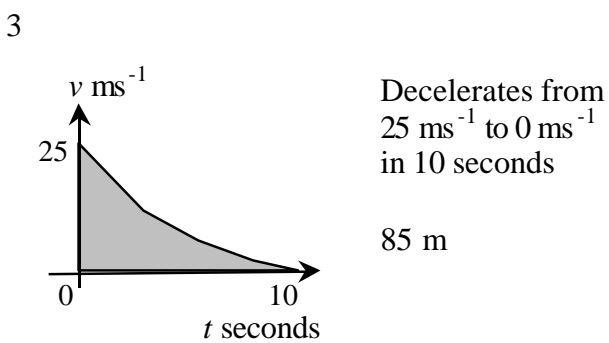
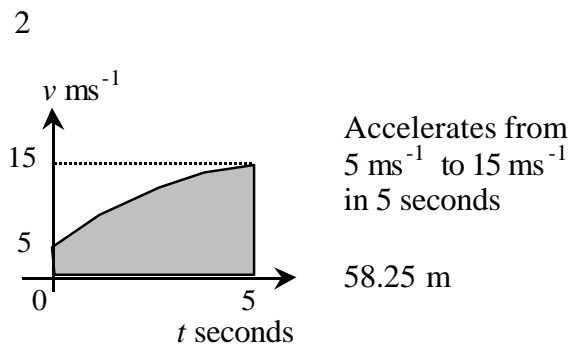
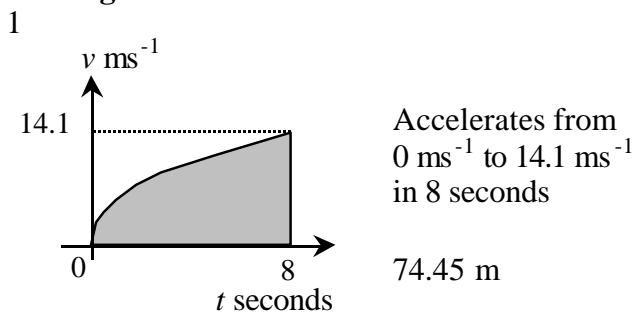


Answers

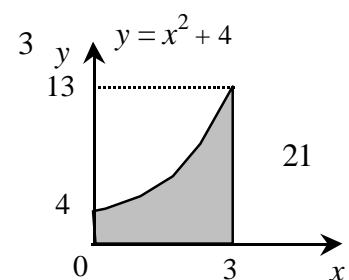
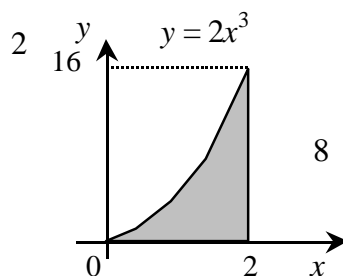
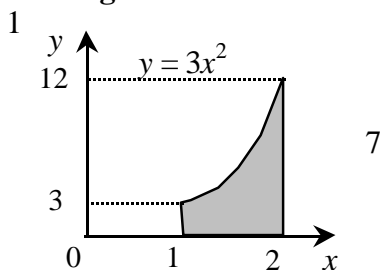
Page 1

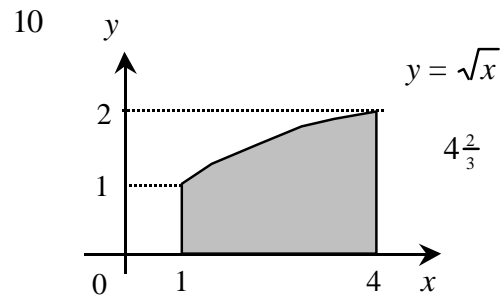
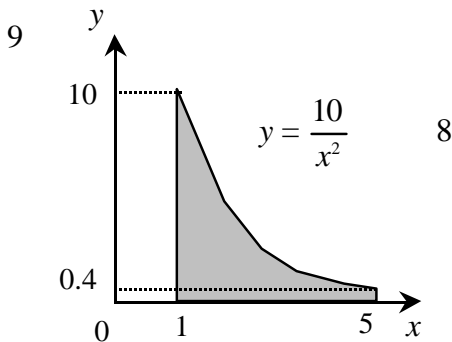
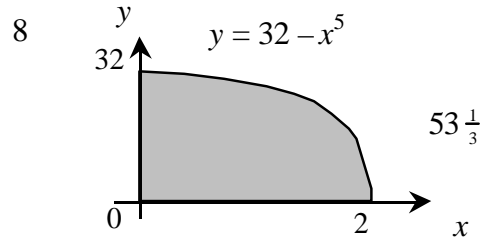
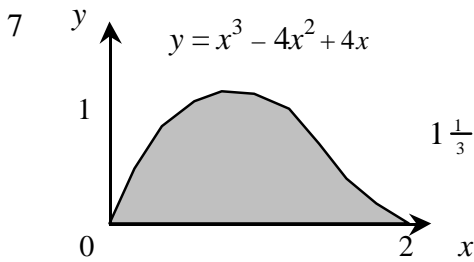
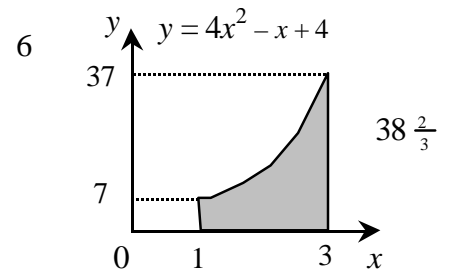
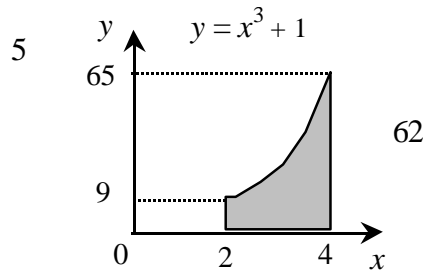
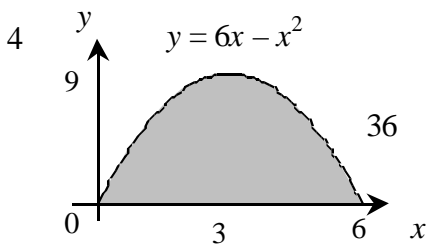


Page 2



Page 4





Page 6

- 1 75.4 m (3sf) 2 $58 \frac{1}{3}$ m 3 $83 \frac{1}{3}$ m 4 72 m
- (Graphs very similar to sketches given for Page 2.)

